MATH2050a Mathematical Analysis I

Exercise 8 suggested Solution

5. Show that the polynomial $p(x) := x^4 + 7x^3 - 9$ has at least two real roots. Use a calculator to locate these roots to within two decimal places.

Solution:

$p(x) := x^4 + 7x^3 - 9$, hence $p(-8) = 503$, $p(-7) = -9$, $p(1) = -1$, $p(2) = 63$. By Intermediate-Value Thm, there exists $a \in (-8, -7)$ and $b \in (1, 2)$ such that $f(a) = f(b) = 0$.

For (-8, -7), we calculate that $p(-7.5) = 201.938 > 0$, hence one root must be contained in (-7.5, -7). we calculate that $p(-7.25) = 86.2695 > 0$, so we just need consider (-7.25, -7). By repeating these steps, finally get the interval $(-7.03125, -7.01563)$,

$$p(-7.03125) = 1.80295 > 0, \quad p(-7.01563) = -3.60466 < 0$$

$$p(\tfrac{-7.03125+(-7.01563)}{2}) = p(-7.02344) = -0.8799.$$

Therefore, $a \approx -7.02344$. Using the same method, we can get $b \approx 1.039$.

3. Use the Nonuniform Continuity Criterion 5.4.2 to show that the following functions are not uniformly continuous on the given sets.

(a) $f(x) := x^2, \quad A := [0, \infty)$.

(b) $g(x) := sin(1/x), \quad B := (0, \infty)$.

Solution:

(a) Let $\{a_n\}$ and $\{b_n\}$ be two sequence defined as $a_n = n + \frac{1}{n}$, and $b_n = n$. Then, $\{\mathrm{a}_n\} \subset A$, $\{\mathrm{b}_n\} \subset A$, and $\lim|a_n - b_n| = 0$.

But $|f(a_n) - f(b_n)| = 2n + \frac{1}{n^2} \geq 2$, $\forall n \in N$. By Nonuniform Continuity Criterion 5.4.2, $f(x) := x^2$ is not uniformly continuous on A.

(b) let $x_n = \frac{1}{n\pi}$, $y_n = \frac{1}{2n\pi+\pi/2}$, $n \in N$. Then, $\{x_n\} \subset B$, $\{y_n\} \subset B$, and $\lim|x_n - y_n| \leq lim(\frac{1}{n\pi} + \frac{1}{2n\pi+\pi/2}) = 0$, hence $\lim|x_n - y_n| = 0$. But $|f(x_n) - f(y_n)| = 1$, $\forall n \in N$. By Nonuniform Continuity Criterion 5.4.2, $g(x) := sin(1/x)$ is not uniformly continuous on B.

4. Show that the function $f(x) := 1/(1+x^2)$ for $x \in R$ is uniformly continuous on R.

Solution:

For any $x, y \in R$,

$$\begin{aligned}|f(x) - f(y)| &= |\tfrac{1}{1+x^2} - \tfrac{1}{1+y^2}| \\ &= \tfrac{|y^2-x^2|}{(1+x^2)(1+y^2)} \\ &= |\tfrac{x+y}{(1+x^2)(1+y^2)}||x-y| \\ &= (\tfrac{1}{1+y^2}\tfrac{x}{1+x^2} + \tfrac{1}{1+x^2}\tfrac{y}{1+y^2})|x-y| \\ &\leq (\tfrac{1}{1+y^2}\tfrac{|x|}{1+x^2} + \tfrac{1}{1+x^2}\tfrac{|y|}{1+y^2})|x-y|\end{aligned}$$

Since $\forall t \in R$, $\frac{|t|}{1+t^2} < 1$ and $\frac{1}{1+t^2} < 1$, we get

$$|f(x) - f(y)| \leq (1 \times 1 + 1 \times 1)|x - y| = 2|x - y|$$

For each $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$, for any $x, y \in R, |x-y| < \delta$, we have $|f(x)-f(y)| \leq 2|x - y| < \epsilon$. Therefore, f is uniformly continuous on R.

7. If $f(x) := x$ and $g(x) := sin(x)$, show that both f and g are uniformly continuous on R, but that their product fg is not uniformly continuous on R.

Solution:

For any $x, y \in R$, $|f(x) - f(y)| = |x - y|$. For each $\epsilon > 0$, choose $\delta = \epsilon$, for any $x, y \in R, |x - y| < \delta$, we have $|f(x) - f(y)| = |x - y| < \epsilon$.Therefore, f is uniformly continuous on R.

For any $x, y \in R$, $|g(x) - g(y)| = |sinx - siny| = |2cos(\frac{x+y}{2})sin(\frac{x-y}{2})|$

$$\leq 2|sin(\tfrac{x-y}{2})| \leq 2|x - y|$$

For each $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$, for any $x, y \in R, |x-y| < \delta$, we have $|g(x)-g(y)| = 2|x - y| < \epsilon$. Therefore, g is uniformly continuous on R.

Let $h(x) = f(x)g(x) = xsinx$. let $x_n = 2n\pi$, $x_n = 2n\pi + \frac{1}{n}$, then $\lim|x_n - y_n| = 0$.

But $|h(x_n) - h(y_n)| = |0 - (2n\pi + \frac{1}{n})sin(\frac{1}{n})|$

$$= |(2n\pi + \tfrac{1}{n})sin(\tfrac{1}{n})|$$

Since lim$nsin(\frac{1}{n}) = 1$, lim $\frac{1}{n})sin(\frac{1}{n}) = 0$, we have $|h(x_n) - h(y_n)| = 1$. Therefore, fg is not uniformly continuous on R.